ALEKSANDROV PROJECTION PROBLEM FOR CONVEX LATTICE SETS

NING ZHANG

Abstract. Let K and L be origin-symmetric convex integer polytopes in $\mathbb{R}^n$. We study a discrete analogue of the Aleksandrov projection problem. If for every $u \in \mathbb{Z}^n$, the sets $(K \cap \mathbb{Z}^n)|u^\perp$ and $(L \cap \mathbb{Z}^n)|u^\perp$ have the same number of points, is then $K = L$? We give a positive answer to this problem in $\mathbb{Z}^2$ under an additional hypothesis that $(2K \cap \mathbb{Z}^2)|u^\perp$ and $(2L \cap \mathbb{Z}^2)|u^\perp$ have the same number of points.

1. Introduction

Let K be a convex body in $\mathbb{R}^n$, i.e. a compact convex set with nonempty interior. We say that K is origin-symmetric if $K = -K$, where $tK := \{tx \in \mathbb{R}^n : x \in K\}, t \in \mathbb{R}$. In 1937, Aleksandrov proved the following result [1]:

Theorem 1.1. *Let $K, L \subset \mathbb{R}^n$ be origin-symmetric convex bodies. If*

$$\mathrm{vol}_{n-1}(K|u^\perp) = \mathrm{vol}_{n-1}(L|u^\perp)$$

for every $u \in S^{n-1}$, then K=L.

Here $u^\perp := \{x \in \mathbb{R}^n : \langle x, u\rangle = 0\}$.

Gardner, Gronchi, and Zong suggested a discrete version of the Aleksandrov projection problem (see [2]). We say A is a convex lattice set if $\mathrm{conv}\,(A) \cap \mathbb{Z}^n = A$, where $\mathrm{conv}\,(A)$ is the convex hull of A.

Problem 1.2. *Let $A, B \subset \mathbb{Z}^n$ be origin-symmetric convex lattice sets. If $|A|u^\perp| = |B|u^\perp|$ for every $u \in \mathbb{Z}^n$, is it true that $A = B$?*

Here, $|A|u^\perp|$ is the cardinality of $A|u^\perp$. Since the convex hull of a convex lattice set is a convex integer polytope, i.e. a polytope all of whose vertices are in $\mathbb{Z}^n$, it would be convenient to restate the problem as follows. Let $K, L \subset \mathbb{R}^n$ be origin-symmetric convex integer polytopes. If $|(K \cap \mathbb{Z}^n)|u^\perp| = |(L \cap \mathbb{Z}^n)|u^\perp|$ for every $u \in \mathbb{Z}^n$, is it true that $K = L$?

In [2], the authors gave a negative answer to Problem 1.2 in $\mathbb{Z}^2$. However, it is not known whether there are other counterexamples. Zhou [6] and Xiong [4] showed that these counterexamples are unique in some special classes. For higher dimensions, this problem is still open. Some work on related problems has been done in [3]. Since the answer is negative in

Author is partially supported by a grant from NSERC.

dimension 2, Gardner, Gronchi, and Zong asked if it is possible to impose reasonable additional conditions to make the answer affirmative. In this paper, we obtain a positive answer to Problem 1.2 in $\mathbb{Z}^2$ under an additional hypothesis.

Before we state the theorem, some definition should be introduced (see [1] and [5]). Let K be a convex body in $\mathbb{R}^n$. The support function of K in the direction u is

$$h_K(u) := \sup\{\langle u, x\rangle : x \in K\}.$$

The width function of K in the direction u is

$$w_K(u) := h_K(u) + h_K(-u).$$

If K is a convex integer polytope, then we denote

$$D_1K := \{u \in Z^n : \exists x_1, x_2 \in K \cap \mathbb{Z}^n, u \parallel x_1x_2\}.$$

For a directed segment u with the initial point $(p_1, \dots, p_n) \in \mathbb{Z}^n$ and the end point $(q_1, \dots, q_n) \in \mathbb{Z}^n$, let

$$\hat{u} := (\frac{q_1 - p_1}{d}, \dots, \frac{q_n - p_n}{d})$$

denote the primitive vector in the direction u, where $d = \gcd(q_1 - p_1, \dots, q_n - p_n)$.

We will need the well-known Pick's theorem (see [5]). Let $K \subset \mathbb{R}^2$ be a convex integer polygon. Then

$$\mathrm{vol}_2(K) = |K \cap \mathbb{Z}^2| - \frac{1}{2}|\partial K \cap \mathbb{Z}^2| - 1,$$

where ∂K is the boundary of K.

We are now ready to state our main result.

Theorem 1.3. *Let $K, L \subset \mathbb{R}^2$ be origin-symmetric convex integer polygons. If*

$$|(K \cap \mathbb{Z}^2)|u^\perp| = |(L \cap \mathbb{Z}^2)|u^\perp|$$

and

$$|(2K \cap \mathbb{Z}^2)|u^\perp| = |(2L \cap \mathbb{Z}^2)|u^\perp|$$

for all $u \in \mathbb{Z}^2$, then $K = L$.

Remark 1.4. *It will be clear from the proof that we do not need projections in all directions, only in directions parallel to the edges of K and L, and one more direction $\xi \in \mathbb{Z}^2 \backslash (D_1K \cup D_1L)$.*

2. proof of Theorem 1.3

Theorem 2.1. *Let K be an origin-symmetric convex integer polygon in $\mathbb{R}^2$ with edges $\{e_i\}_{i=1}^{2n}$, where e_i and e_{n+i} are symmetric with respect to the origin. Then*

$$|(K \cap \mathbb{Z}^2)|e_i^\perp| = |\hat{e}_i| w_K(e_i^\perp) + 1, \text{ for } 1 \le i \le n,$$

where $|\hat{e}_i|$ is the length of the primitive vector parallel to e_i. Here and below, $w_K(u^\perp)$ means the width in the direction perpendicular to u.

We will first prove the theorem in a simple case.

Lemma 2.2. *Let $K \subset \mathbb{R}^2$ be a parallelogramm with edges $\{e_i\}_{1\le i\le 4}$, where $e_1 \parallel e_3$ and $e_2 \parallel e_4$. Then*

$$|(K\cap\mathbb{Z}^2)|e_i^{\perp}| = |\hat{e}_i| w_K(e_i^{\perp}) + 1, \text{ for } i = 1,2.$$

Proof. Consider the point lattice Λ generated by $\hat{e}_1$ and $\hat{e}_2$ and the quotient map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$. Set $l(e_1)$ to be the line passing through the origin and parallel to e_1. If $x \in K \cap \Lambda$, then

$$|(x + l(e_1)) \cap (K \cap \mathbb{Z}^2)| = |e_1 \cap \mathbb{Z}^2|.$$

If $x \in (K \cap \mathbb{Z}^2)\backslash\Lambda$, then $\pi((x + l(e_1)) \cap (K \cap \mathbb{Z}^2))$ contains only one point; otherwise, $x \in \Lambda$. Thus,

$$|(x + l(e_1)) \cap (K \cap \mathbb{Z}^2)| = |e_1 \cap \mathbb{Z}^2| - 1.$$

One can see that,

$$|(K \cap \Lambda)|e_1^{\perp}| = |e_2 \cap \mathbb{Z}^2|.$$

Furthermore when projecting $(K\cap\mathbb{Z}^2)\backslash\Lambda$ onto $e_1^{\perp}$, each point in the projection has $|e_1\cap\mathbb{Z}^2|-1$ preimages. Thus,

$$|((K \cap \mathbb{Z}^2)\backslash\Lambda)|{e_1}^{\perp}| = \frac{|(K\cap\mathbb{Z}^2)| - |e_1\cap\mathbb{Z}^2||e_2\cap\mathbb{Z}^2|}{|e_1\cap\mathbb{Z}^2| - 1};$$

hence,

$$\begin{aligned}
|(K\cap\mathbb{Z}^2)|e_1^{\perp}| &= \frac{|(K\cap\mathbb{Z}^2)| - |e_1\cap\mathbb{Z}^2||e_2\cap\mathbb{Z}^2|}{|e_1\cap\mathbb{Z}^2| - 1} + |e_2\cap\mathbb{Z}^2| \\
&= \frac{|(K\cap\mathbb{Z}^2)| - |e_2\cap\mathbb{Z}^2|}{|e_1\cap\mathbb{Z}^2| - 1} \\
&= \frac{\mathrm{vol}_2(K) + |e_1\cap\mathbb{Z}^2| - 1}{|e_1\cap\mathbb{Z}^2| - 1} \text{ (by Pick's theorem)} \\
&= \frac{|\hat{e}_1|(|e_1\cap\mathbb{Z}^2| - 1)w_K(e_1^{\perp}) + |e_1\cap\mathbb{Z}^2| - 1}{|e_1\cap\mathbb{Z}^2| - 1} \\
&= |\hat{e}_1| w_K(e_1^{\perp}) + 1.
\end{aligned}$$

□

Proof of Theorem 2.1. Without loss of generality, we only need to compute $|(K \cap \mathbb{Z}^2)|e_1^{\perp}|$. Create a convex lattice set with convex hull being a parallelogramm with edges e_1 and e_{n+1}, denoted by $\varSigma$. Note that, for any $x \in (K \cap \mathbb{Z}^2)\backslash\varSigma$, $x + l(e_1) \cap \varSigma \neq \emptyset$. Thus, there exists $m \in \mathbb{Z}$, such that $x \in \varSigma + me_1$, which implies $x - me_1 \in \varSigma \cap \mathbb{Z}^2$. Therefore, by Lemma 2.2

$$|(K\cap\mathbb{Z}^2)|e_1^{\perp}| = |(\varSigma\cap\mathbb{Z}^2)|e_1^{\perp}| = |\hat{e}_1| w_K(e_1^{\perp}) + 1.$$

□

Theorem 2.1 implies that if K and L have parallel edges, then there is a uniqueness in Problem 1.2.

Lemma 2.3. *Let K be an origin-symmetric convex integer polygon in $\mathbb{Z}^2$. Let $u \in D_1 K$. If $2(|(K\cap\mathbb{Z}^2)|u^\perp| - 1) = |(2K\cap\mathbb{Z}^2)|u^\perp| - 1$, then*

$$|(K\cap\mathbb{Z}^2)|u^\perp| = |\hat{u}|w_K(u^\perp) + 1.$$

Proof. Let $u \in D_1 K$. If u is parallel to one of the edges of K, then, by Theorem 2.1, $|(K\cap\mathbb{Z}^2)|u^\perp| = |\hat{u}|w_K(u^\perp)+1$; if not, consider the pair of points $(x_1,x_2) \in \{(x,y)\in K\times K : xy \parallel \hat{u}\}$ such that

$$\operatorname{dist}(O,\overline{x_1x_2}) = \max_{\{(x,y)\in K\times K: xy\parallel\hat{u}\}} \operatorname{dist}(O,\overline{xy}).$$

Here, we denoted by $\operatorname{dist}(O,A) = \inf_{x\in A}\|x-O\|_2$, the distance between O and a set A. The set $\{(x,y)\in K\times K : xy\parallel\hat{u}\}$ is not empty, since $u\in D_1K$.

Thus, the lines passing through x_1, x_2 and $-x_1, -x_2$ divide $\mathbb{R}^2$ into three parts E_1, E_2, and E_3, where $O\in E_2$ and E_1, E_3 are reflections of each other with respect to O.

Note that, $E_2\cap K\cap\mathbb{Z}^2$ is a convex lattice set and $x_1, x_2, -x_1, -x_2$ lie on two parallel edges of $E_2\cap K$. (Here, $E_2\cap K$ can be a segment.) Then, by Theorem 2.1, we have

$$|(E_2\cap K\cap\mathbb{Z}^2)|u^\perp| = |\hat{u}|w_{E_2\cap K}(u^\perp)+1$$

and set $|(E_1\cap K\cap\mathbb{Z}^2)|u^\perp| = |(E_3\cap K\cap\mathbb{Z}^2)|u^\perp| = m$. We have

$$|(K\cap\mathbb{Z}^2)|u^\perp| = 2m + |\hat{u}|w_{E_2\cap K}(u^\perp) - 1.$$

On the other hand, $|(2E_2\cap 2K\cap\mathbb{Z}^2)|u^\perp| = 2|\hat{u}|w_{E_2\cap K}(u^\perp)+1$. Moreover, a line l parallel to u divides $2E_1\cap 2K$ into two parts of equal width in the direction perpendicular to u, denoted by $E_{11}\cap 2K$ and $E_{12}\cap 2K$, where $\operatorname{dist}(O,E_{11}) > \operatorname{dist}(O,E_{12})$.

Note that there exists a pair of points $y_1, y_2 \in l\cap 2K\cap\mathbb{Z}^2$. To see this, pick a point z from $E_1\cap K\cap\mathbb{Z}^2$ such that $w_{[-z,z]}(u^\perp) = w_K(u^\perp)$, where $[-z,z]$ is the segment connecting $-z$ and z. Then $2z, 2x_1, 2x_2 \in 2K\cap\mathbb{Z}^2$ implies $y_1 = z+x_1, y_2 = z+x_2 \in 2K\cap l$.

Now we obtain $E_{11}\cap 2K \supset E_1\cap K + z$. To see this, assume $E_1\cap K = \{x\in\mathbb{R}^2 : \langle x, v_i\rangle \le a_i\}$ with $x_1, x_2 \in \{x\in\mathbb{R}^2 : \langle x, v_1\rangle = a_1\}$, then $\langle z, v_1\rangle = a_1 - w_{E_1\cap K}(v_1)$ and $\langle u, v_1\rangle = 0$. Thus for any $x\in E_1\cap K$, $\langle x+z, v_i\rangle \le 2a_i$ and $\langle x+z, v_1\rangle \le 2a_1 - w_{E_1\cap K}(v_1)$, implying $x+z \in 2(E_1\cap 2K)\cap E_{11} = E_{11}\cap 2K$.

Since $E_{12}\cap 2K$ contains a parallelogramm $\varparallelogram$ with vertices $2x_1, x_1+x_2, y_1, y_2$, we have

$$|(E_{11}\cap 2K\cap\mathbb{Z}^2)|u^\perp| \ge |(E_1\cap K\cap\mathbb{Z}^2)|u^\perp| = m$$

and

$$|(E_{12}\cap 2K\cap\mathbb{Z}^2)|u^\perp| = |\hat{u}|w_{\varparallelogram}(u^\perp)+1 = |\hat{u}|w_{E_1\cap K}(u^\perp)+1.$$

Hence,

$$|(2E_1\cap 2K\cap\mathbb{Z}^2)|u^\perp| \ge m + |\hat{u}|w_{E_1\cap K}(u^\perp).$$

Therefore,

$$|(2K\cap\mathbb{Z}^2)|u^\perp| \ge 2(m+|\hat{u}|w_{E_1\cap K}(u^\perp)) + 2|\hat{u}|w_{E_2\cap K}(u^\perp) - 1.$$

By the assumption, we have

$$\begin{aligned}
&2(2m+|\hat{u}|w_{E_2\cap K}(u^\perp)-2) = 2(|(K\cap\mathbb{Z}^2)|u^\perp|-1)\\
&= |(2K\cap\mathbb{Z}^2)|u^\perp| - 1 \ge 2(m+|\hat{u}|w_{E_1\cap K}(u^\perp)) + 2|\hat{u}|w_{E_2\cap K}(u^\perp) - 1,
\end{aligned}$$

which implies

$$m \geq |\hat{u}| w_{E_1 \cap K}(u^\perp) + 1.$$

On the other hand, $m \leq |\hat{u}| w_{E_1 \cap K}(u^\perp) + 1$, by constructing a large parallelogramm containing $E_1 \cap K$, that has two edges parallel to u and whose width perpendicular to u is $w_{E_1 \cap K}(u^\perp)$; thus,

$$m = |\hat{u}| w_{E_1 \cap K}(u^\perp) + 1.$$

The conclusion follows. $\square$

Definition 2.4. *Let $\mathcal{K}^n$ be the collection of all origin-symmetric convex bodies in $\mathbb{R}^n$. Define an operator $\Cup : \mathcal{K}^n \times \mathcal{K}^n \to \mathcal{K}^n$, satisfying*

$$A \Cup B := \operatorname{conv}(A \cup B).$$

One can easily prove the following properties.

Proposition 2.5. *Let $A, B \in \mathcal{K}^n$. Then*

$$h_{A \Cup B}(u) = \max\{h_A(u), h_B(u)\} \quad \text{and} \quad w_{A \Cup B}(u) = \max\{w_A(u), w_B(u)\}.$$

Lemma 2.6. *Let K and L be origin-symmetric convex polygons in $\mathbb{R}^2$. If $w_K(u^\perp) = w_L(u^\perp)$ for all $u \in E_K \cup E_L$, then $K = L$. Here, E_K is the collection of all directions parallel to the edges of K.*

Proof. Clearly, $K \subseteq K \Cup L$ and $w_K(u^\perp) = w_{K \Cup L}(u^\perp)$, for all $u \in E_K$. Assume $K \subsetneqq K \Cup L$. Then there exists a point $\{x\} \in K \Cup L$, but $\{x\} \notin K$. Thus we can find a direction $\eta \in E_K$, such that, $2\langle x, \eta\rangle = \langle x - (-x), \eta\rangle > w_K(\eta^\perp)$, implying $w_K(\eta^\perp) < w_{[-x,x] \Cup K}(\eta^\perp)$. On the other hand, since $[-x, x] \Cup K \subseteq K \Cup L$, we have

$$w_{[-x,x] \Cup K}(\eta^\perp) \leq w_{K \Cup L}(\eta^\perp) = w_K(\eta^\perp),$$

Contradiction. $\square$

Proof of Theorem 1.3. Here, we use the weaker condition mentioned in Remark 1.4. Note that, $|(K \cap \mathbb{Z}^2)|u^\perp| < |K \cap \mathbb{Z}^2|$, if $u \in D_1K$; but $|(K \cap \mathbb{Z}^2)|u^\perp| = |K \cap \mathbb{Z}^2|$, if $u \in \mathbb{Z}^2 \backslash D_1K$. For any $u \in E_K$, we have $u \in D_1L$; otherwise,

$$|(L \cap \mathbb{Z}^2)|u^\perp| = |L \cap \mathbb{Z}^2| = |(L \cap \mathbb{Z}^2)|\xi^\perp| = |(K \cap \mathbb{Z}^2)|\xi^\perp| = |K \cap \mathbb{Z}^2| > |(K \cap \mathbb{Z}^2)|u^\perp|$$

for some $\xi \in \mathbb{Z}^2 \backslash (D_1K \cup D_1L)$. Then by Lemma 2.3, we have

$$|(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}| w_K(u^\perp) + 1 \quad \text{and} \quad |(2K \cap \mathbb{Z}^2)|u^\perp| = 2|\hat{u}| w_K(u^\perp) + 1.$$

By the assumption,

$$\begin{aligned} |(2L \cap \mathbb{Z}^2)|u^\perp| - 1 &= |(2K \cap \mathbb{Z}^2)|u^\perp| - 1 = 2|\hat{u}| w_K(u^\perp) \\ &= 2(|(K \cap \mathbb{Z}^2)|u^\perp| - 1) = 2(|(L \cap \mathbb{Z}^2)|u^\perp| - 1). \end{aligned}$$

Applying Lemma 2.3,

$$|(L \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}| w_L(u^\perp) + 1 = |(K \cap \mathbb{Z}^2)|u^\perp| = |\hat{u}| w_K(u^\perp) + 1.$$

Therefore,

$$w_L(u^{\perp}) = w_K(u^{\perp}),$$

for any $u \in E_K$. Similarly, we can show $w_L(u^{\perp}) = w_K(u^{\perp})$, for any $u \in E_L$. Then the conclusion follows from Lemma 2.6. $\square$

3. Acknowledgement

I would like to express my gratitude to my supervisor, Dr. Vladyslav Yaskin, for fruitful discussions.

References

[1] R. Gardner, *Geometric tomography,* second edition, Cambridge University Press, New York, 2006.
[2] R. Gardner, P. Gronchi, and C. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram*, Discrete Comput. Geom., **34** (2005) 391–409.
[3] D. Ryabogin, V. Yaskin, and N. Zhang, *Unique determination of convex lattice sets*, preprint.
[4] H. Xiong, *On a discrete version of Alexandrov's projection theorem,* Acta Mathematica Sinica, English Series, (8) **29** (2013), 1597–1606.
[5] G. Ziegler, *Lectures on polytopes*, Springer-Verlag, Berlin, 1995.
[6] J. Zhou, *On the projections of convex lattice sets*, Acta Mathematica Sinica, English Series, (10) **26** (2010), 1969–1980.

Ning Zhang, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton AB, T6G 2G1, Canada
E-mail address: nzhang2@ualberta.ca